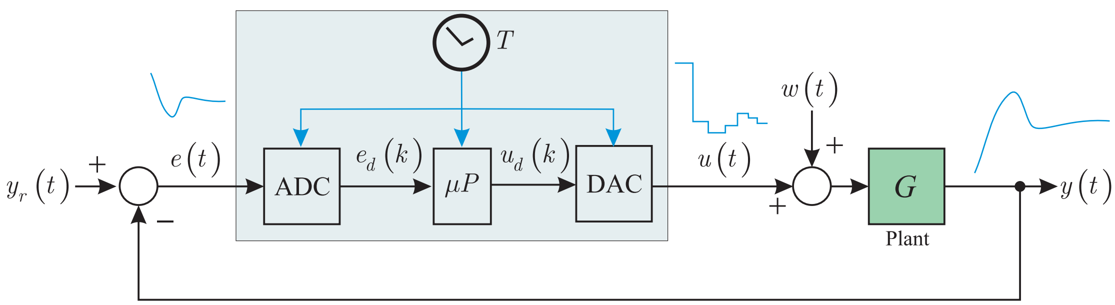
Lecture Notes

Introduction to Digital Control of Dynamic Systems

And Recipes for Engineers



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Chapter 1
Introduction

1.1 What is Digital Control?

Automatic control is the science that develops techniques to steer, guide, control dynamic systems. These systems are built by humans and must perform a specific task. Examples of such dynamic systems are found in biology, physics, robotics, finance, etc.

Digital Control means that the control laws are implemented in a digital device, such as a microcontroller or a microprocessor. Such devices are light, fast and economical.

The points that will be examined in these lecture notes are the following:

- transformation of an already designed continuous-time controller into a discrete-time controller,
- discretization of continuous systems,
- direct synthesis of discrete-time control systems,
- practical considerations and precautions when implementing a digital controller.

1.2 Examples of Digitally Controlled Systems

Nowadays, digitally controlled systems are everywhere,

- automotive industry: speed regulators in cars,
- aeronautic/space industry: autopilots, automatic take off/landing, cruise control
- chemistry: pharmaceutical industries, oil transformation, liquid level in tanks
- robotics: robot-arm trajectory control, manipulation,
- housing: in-house temperature regulation

1.3 Classical Control System Architectures

1.3.1 Definitions

Fig. 1.1 shows a classical control architecture, where the following key elements are described below:

- the variable to be controlled is y,
- the reference variable is y_c, the control objective is thus to have y follow y_c,
- the plant transfer function is G,
- the controller transfer function is K,
- the error signal is $e = y_c - y$,

Chapter 2
Z-Transform

Objectives of this chapter:

- Introduce the z-transform and its main properties.
- Calculate the z inverse transform.
- Define the transfer function of a discrete linear causal stationary system at rest.

2.1 Introduction

A key tool in the theory of discrete systems is the ***z-transform***. In fact, the z-transform plays a role in discrete systems that is similar to the role of the Laplace transform in the theory of analog systems. In fact, this equivalence is all the more visible if the following relation between $z-$variable and the $s-$variable holds:

$$z \equiv e^{sT}. \tag{2.1}$$

Let us, first of all, introduce the $z-$transform and then demonstrate the above relationship.

2.2 Definitions

2.2.1 z− Transform

The $z-$transform of a discrete signal $\{w(kT)\}$, denoted by $W(z)$ or $\mathcal{Z}\{w(kT)\}$ is defined by:

$$W(z) = \mathcal{Z}\{w(kT)\} = \sum_{0}^{\infty} w(kT) z^{-k} \ , \tag{2.2}$$

where z is a complex variable. The discrete samples are supposed to be null for all $k < 0$:

$$w(kT) = 0, \quad k < 0 \tag{2.3}$$

Radius of Convergence

If there exists a number $r \in \Re^{+}$, such that:

- for all $z \ / \ |z| > r$, the sum converges,

Chapter 3
Controller Emulation

In this chapter, it is shown how to obtain a discrete-time controller by *emulation*, i.e. converting a continuous-time controller into a discrete-time controller using the method of path "B" shown in Fig. 3.1.

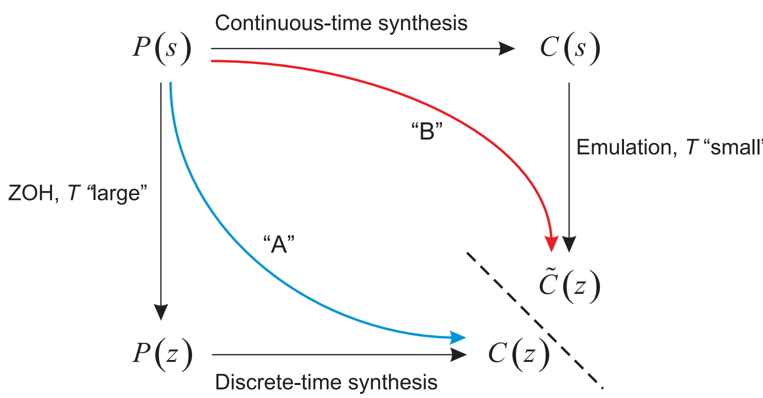

Fig. 3.1 Two controller design approaches: path "A" corresponds to direct discrete-time controller synthesis, whereas path "B" corresponds to the "Emulation" process.

3.1 Introductory Example of Controller Emulation

The Proportional-Integral (PI) controller is well known and the most used control technique in industry. In the continuous-time domain, this controller is defined by:

$$u(t) = K_p \left(e(t) + \frac{1}{T_i} \int_0^t e(\tau) d\tau \right), \tag{3.1}$$

This controller contains parameters K_p and T_i that are to be tuned properly to achieve satisfactory response of the controlled system. There exists a large number of tuning techniques in the literature. The control error is $e(t) = y_c(t) - y(t)$, the reference signal is $y_c(t)$, the variable to be controlled is $y(t)$, and the controller output signal is $u(t)$.

Now, consider the discrete time instant $t_k = kT$, with $T \in \Re^+$, equation 3.1 is re-written at time $t = t_k = kT$

$$u(kT) = K_p \left(e(kT) + \frac{1}{T_i} \int_0^{kT} e(\tau) d\tau \right) , \tag{3.2}$$

and the integral term is approximated by the method of *rectangles* as shown in Fig. 3.2 by

$$q(kT) = \sum_{i=0}^{k-1} e(iT)T \ . \tag{3.3}$$

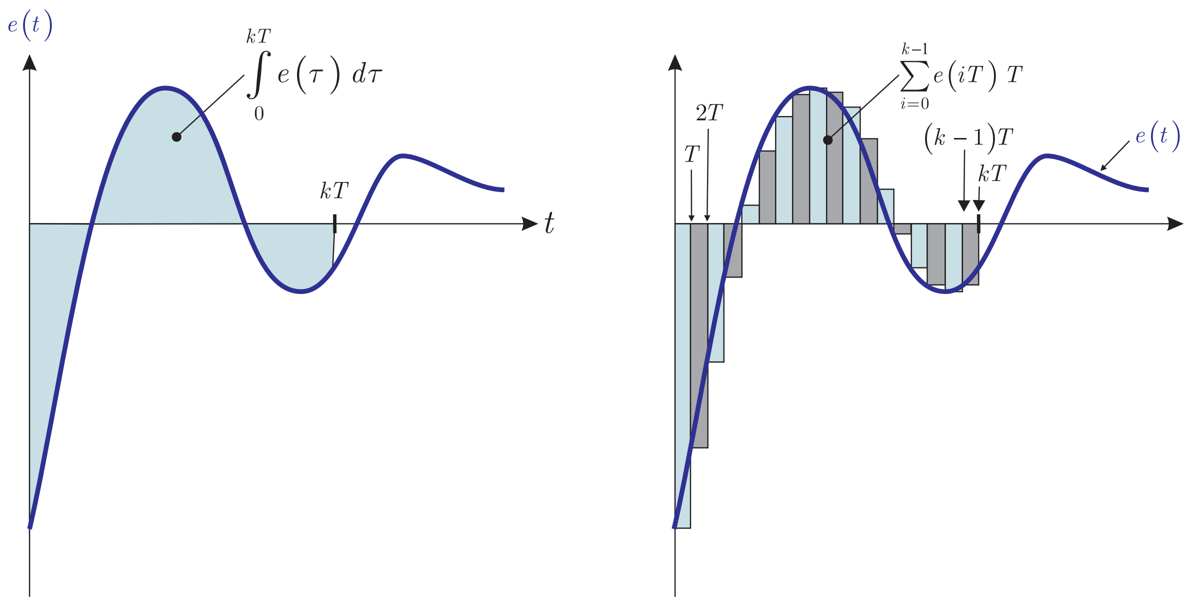

Fig. 3.2 Approximation of the integral term by the method of rectangles

Equation 3.2 thus becomes

$$u(kT) = K_p \left(e(kT) + \frac{1}{T_i} \sum_{i=0}^{k-1} e(iT)T \right) \ . \tag{3.4}$$

The integral term can be expressed in a recursive form as follows:

$$q(kT+T) = \sum_{i=0}^{k} e(iT)T = \sum_{i=0}^{k-1} e(iT)T + e(kT)T = q(kT) + e(kT)T, \tag{3.5}$$

therefore the digital PI controller can be summarized as:

1. Compute the error term: $e(kT) = y_c(kT) - y(kT)$
2. Update the integral term: $q(kT) = q\left((k-1)T\right) + T \cdot e\left((k-1)T\right)$
3. Update controller output: $u(kT) = K_p \left(e(kT) + \frac{1}{T_i} q(kT) \right)$
4. Store in memory $q(kT)$ and $e(kT)$ as they will be used at the next iteration and will become $q\left((k-1)T\right)$ and $e\left((k-1)T\right)$, respectively.

Remarks:

- In order that the digital controller produces a closed-loop behavior similar to the one obtained in continuous time, the sampling period T must be chosen sufficiently small.

- If the PI controller is designed only with continuous-time considerations, i.e., the DAC and ADC are not considered, a decrease in control performance is expected with the digital controller compared to the continuous-time (analog) controller. The stability of the system can no longer be guaranteed. It is only possible to predict the performance of a digital controller, in terms of response and stability, given a certain sampling period T, when the discrete nature of the DAC and ADC is explicitly taken into account in the controller design. These aspects will be discussed later in this lecture.

3.2 Shift Operators

3.2.1 Definition

Let us introduce the discrete-time Heaviside operator z.

- The notation $x(k)$ denotes the value of the variable $x(t)$ at time $t = kT$.
- The operator z is used to denote a forward shift by one sampling interval, i.e.,

$$z \cdot x(k) = x(k+1)\ . \tag{3.6}$$

- The operator z^{-1} denotes a backward shift operation by one sampling interval, i.e.,

$$z^{-1} \cdot x(k) = x(k-1)\ . \tag{3.7}$$

Example of use: Consider the following continuous-time equation:

$$y\left((k+n)T\right) + a_{n-1} \cdot y((k+n-1)T) + \cdots + a_0 \cdot y(kT) = b_{n-1} \cdot u\left((k+n-1)T\right) + \cdots + b_0 \cdot u(kT)\ , \tag{3.8}$$

which can be simplified by dropping the time constant T as:

$$y(k+n) + a_{n-1} \cdot y(k+n-1) + \cdots + a_0 \cdot y(k) = b_{n-1} \cdot u(k+n-1) + \cdots + b_0 \cdot u(k)\ . \tag{3.9}$$

Using the shift operator z, the discrete representation of the above linear difference equation is obtained as follows:

$$\begin{aligned} z^n \cdot y(k) + a_{n-1} \cdot z^{n-1} \cdot y(k) + \cdots + a_0 \cdot y(k) &= b_{n-1} \cdot z^{n-1} \cdot u(k) + \cdots + b_0 \cdot u(k) \\ y(k) + a_{n-1} \cdot z^{-1} \cdot y(k) + \cdots + a_0 \cdot z^{-n} \cdot y(k) &= b_{n-1} \cdot z^{-1} \cdot u(k) + \cdots + b_0 \cdot z^{-n} \cdot u(k) \quad . \end{aligned} \tag{3.10}$$

A transfer function from the system input u to the system output y can be obtained as a discrete form as follows:

$$y(k) = \frac{b_{n-1} \cdot z^{-1} + \cdots + b_0 \cdot z^{-n}}{1 + a_{n-1} \cdot z^{-1} + \cdots + a_0 \cdot z^{-n}} \quad u(k)\ . \tag{3.11}$$

3.3 Discretization Methods

Let us suppose that a continuous-time controller $K(s)$ has been designed and is given in the form of a transfer function in the Laplace domain as follows:

$$K(s) = \frac{U(s)}{E(s)} = \frac{b_0 s^m + \cdots + b_{m-2}s^2 + b_{m-1}s + b_m}{s^n + \cdots + a_{n-2}s^2 + a_{n-1}s + a_n}, \qquad \text{with} \quad \mathrm{n} \geq \mathrm{m} \tag{3.12}$$

It is intended to find a technique to transform such a transfer function expressed with the Laplace variable s into a discrete equivalent representation with the z variable. There exist three different methods to

Chapter 4
Sample-and-hold Devices

4.1 Introduction

In practice, the discrete control signal $u_d(k)$ generated by the processor at the time instant $t_k = kT$, enters the real continuous-time system to be controlled. Its output is sampled and yields the discrete sequence of measurement data $y(k)$. Thus, the block in the gray box in Fig. 4.1 can be considered as a "purely" discrete system to control by the discrete control system (outside the gray box in Fig. 4.1).

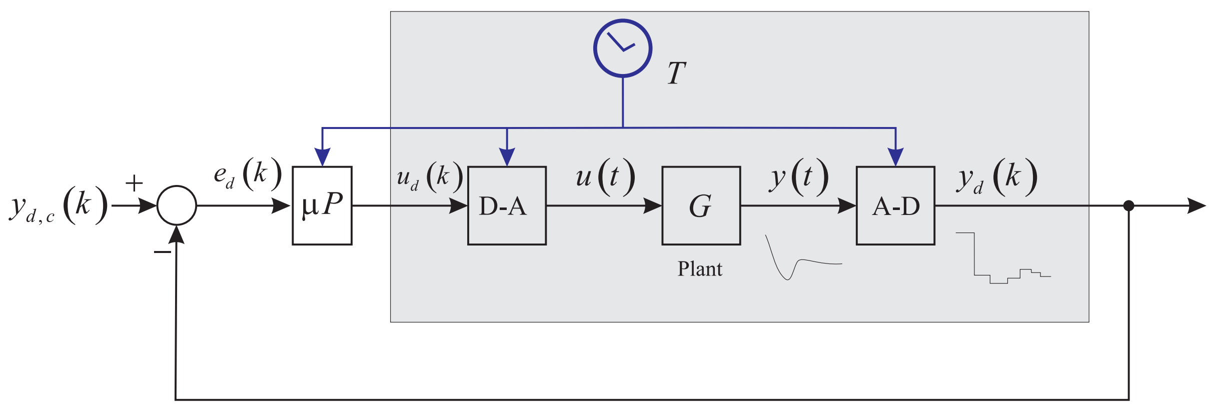

Fig. 4.1 Sampling of the process to control

4.2 Discrete Transfer Function of the Sampled System to Control

Consider a continuous-time linear system with a transfer function $G(s)$. The system composed of the digital-to-analog converter, the time continuous plant and the analog-to-digital converter is a discrete process with a transfer function as follows:

$$H(z) = (1 - z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)\right\} \tag{4.1}$$

See Fig. 4.2

The transfer function $H(z) = \frac{Y(z)}{U(z)}$ is independent on $U(z)$, therefore, in order to compute $H(z)$ one may choose $U(z)$ to its convenience, evaluate $Y(z)$ and finally form the quotient $\frac{Y(z)}{U(z)}$. Let us follow this 3-step procedure:

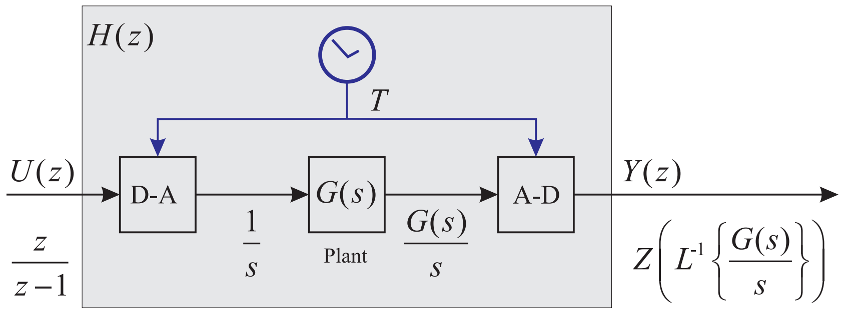

Fig. 4.2 Discrete description of the system to control

1. Selection of an input signal: consider the discrete unit-step signal described by its sequence $\{u(kT)\} = \{\dots, 0, 1, 1, \dots\}$. Its $z-$transform is obtained as $U(z) = \frac{z}{z-1}$.
2. Evaluation of the output:

   - if the input of the *digital-to-analog* converter is the discrete unit-step signal, then the output of the DAC is an analog unit-step signal.
   - The Laplace transform of a unit-step input is $\frac{1}{s}$.
   - The Laplace transform of the analog plant's output is $\frac{G(s)}{s}$. Equivalently, in the time domain the plant's output signal is $y(t) = \mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)$
   - The output of the *analog-to-digital* converter is the sampled version of $y(t)$, its $z-$transform is obtained as $Y(z) = \mathcal{Z}\left(\{y(kT)\}\right) = \mathcal{Z}\left(\left\{\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)\right\}\right)$

3. Forming the transfer function $H(z) = \frac{Y(z)}{U(z)}$:

$$H(z) = \frac{Y(z)}{U(z)} = \frac{\mathcal{Z}\left(\left\{\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)\right\}\right)}{(1-z^{-1})^{-1}} = (1-z^{-1})\mathcal{Z}\left(\left\{\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)\right\}\right) \tag{4.2}$$

4.3 Commonly Used Transfer Functions

Usually, the output y of a system to control has to follow a reference signal y_c in the presence of a disturbance signal w at input of the plant to control, as shown in Fig. 4.3.

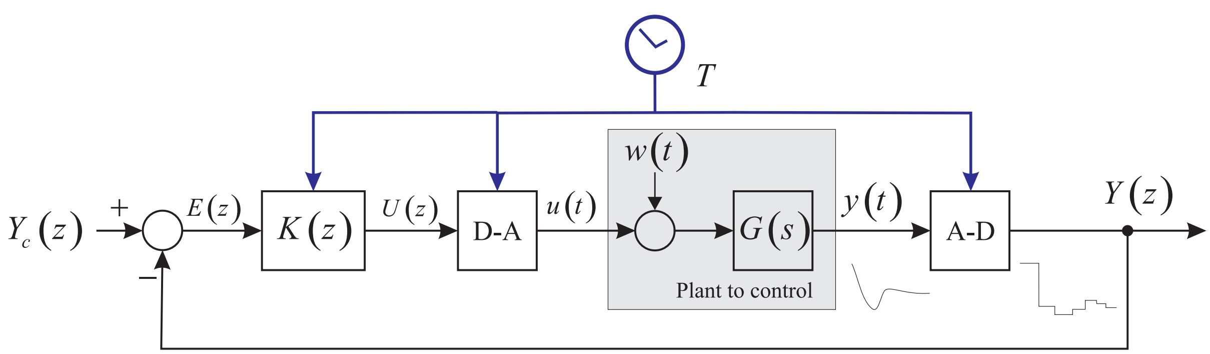

Fig. 4.3 Discrete system and input disturbance

$$
\begin{aligned}
\boldsymbol{x}[k+1] &= e^{\boldsymbol{A}(k+1)T_s}\boldsymbol{x}(0) + \int_0^{(k+1)T_s} e^{\boldsymbol{A}((k+1)T_s-\sigma)}\boldsymbol{B}\boldsymbol{u}(\sigma)d\sigma \\
&\quad + \int_0^{(k+1)T_s} e^{\boldsymbol{A}((k+1)T_s-\sigma)}\boldsymbol{w}(\sigma)d\sigma \ , \\
&= \int_{kT_s}^{(k+1)T_s} e^{\boldsymbol{A}((k+1)T_s-\sigma)}\boldsymbol{B}\boldsymbol{u}(\sigma)d\sigma + \int_{kTs}^{(k+1)T_s} e^{\boldsymbol{A}((k+1)T_s-\sigma)}\boldsymbol{w}(\sigma)d\sigma \\
&\quad + e^{\boldsymbol{A}T_s}\left[e^{\boldsymbol{A}kT_s}\boldsymbol{x}(0) + \int_0^{kT_s} e^{\boldsymbol{A}(kT_s-\sigma)}\boldsymbol{B}\boldsymbol{u}(\sigma)d\sigma + \int_0^{kT_s} e^{\boldsymbol{A}(kT_s-\sigma)}\boldsymbol{w}(\sigma)d\sigma\right] .
\end{aligned}
\tag{4.14}
$$

The bracketed term in (4.14) is actually $\boldsymbol{x}[k] = \boldsymbol{x}(kT_s)$, and the second and third terms can be simplified by using the substitution $\tau = (k+1)T_s - \sigma$ as follows:

$$
\begin{aligned}
\int_{kT_s}^{(k+1)T_s} e^{\boldsymbol{A}((k+1)T_s-\sigma)}\boldsymbol{B}\boldsymbol{u}(\sigma)d\sigma &= -\int_{T_s}^{0} e^{\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}((k+1)T_s-\sigma)d\tau \ , \\
&= \int_0^{T_s} e^{\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}((k+1)T_s-\sigma)d\tau \ .
\end{aligned}
\tag{4.15}
$$

Assuming also that the input $\boldsymbol{u}$ is constant over the integration interval (zero-order hold) yields

$$
\boldsymbol{u}((k+1)T_s-\sigma) = \boldsymbol{u}(kT_s) = \boldsymbol{u}[k] \tag{4.16}
$$

and therefore, the discrete model is finally obtained as:

$$
\begin{aligned}
\boldsymbol{x}[k+1] &= e^{\boldsymbol{A}T_s}\boldsymbol{x}[k] + \left(\int_0^{T_s} e^{\boldsymbol{A}\tau}d\tau\right)\boldsymbol{B}\boldsymbol{u}[k] + \left(\int_0^{T_s} e^{\boldsymbol{A}\tau}\boldsymbol{w}(\tau)d\tau\right) \ , \\
&= \boldsymbol{A_d}\boldsymbol{x}[k] + \left(\int_0^{T_s}\boldsymbol{\Phi}(\tau)d\tau\right)\boldsymbol{B}\boldsymbol{u}[k] + \left(\int_0^{T_s}\boldsymbol{\Phi}(\tau)\boldsymbol{w}(\tau)d\tau\right) \ , \\
&= \boldsymbol{A_d}\boldsymbol{x}[k] + \boldsymbol{B_d}\boldsymbol{u}[k] + \boldsymbol{w_k} \ .
\end{aligned}
\tag{4.17}
$$

Clearly, it appears that the discrete transition ($\boldsymbol{A_d}$) and input ($\boldsymbol{B_d}$) matrices are obtained as

$$
\begin{aligned}
\boldsymbol{A_d} &= e^{\boldsymbol{A}T_s} \\
\boldsymbol{B_d} &= \left(\int_0^{T_s}\boldsymbol{\Phi}(\tau)d\tau\right)\boldsymbol{B}
\end{aligned}
\tag{4.18}
$$

The discrete fundamental or transition matrix can be found by evaluating the continuous transition matrix at the sampling time T_s or $\boldsymbol{A_d} = \boldsymbol{\Phi}(T_s)$. As demonstrated in several practical examples in the book by Zarchan and Musoff[1], the transition matrix can be approximated by taking only the first two elements of the Taylor series. Adding higher-order terms does not bring significant improvements in the performance of the filter. Therefore, the discrete transition matrix is computed as follows:

$$
\boldsymbol{A_d} \approx \boldsymbol{I} + \boldsymbol{A}T_s \ , \tag{4.19}
$$

where the continuous-time system dynamics matrix is $\boldsymbol{A}$.

[1] P. Zarchan and H. Musoff. *Fundamentals of Kalman Filtering: A Practical Approach, Second Edition.* Volume 208, Progress in Astronautics and Aeronautics, AIAA Inc., Reston, VA, 2005.